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A priori error analysis for state constrained boundary control problems. Part I: control discretization

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Abstract. This is the first of two papers concerned with a state-constrained optimal control problems with boundary control, where the state constraints are only imposed in an interior subdomain. We apply the virtual control concept introduced in [20] to regularize the problem. The arising regularized optimal control problem is discretized by finite elements and linear and continuous ansatz functions for the boundary control. In the first part of the work, we investigate the errors induced by the regularization and the discretization of the boundary control. The second part deals with the error arising from discretization of the PDE. Since the state constraints only appear in an inner subdomain, the obtained order of convergence exceeds the known results in the field of a priori analysis for state-constrained problems.

1. Introduction. This is the first of two papers dealing with the following optimal control problem with Neumann boundary control and pointwise state and control constraints:

$$\left.\begin{aligned} \min \quad J(y,u) &:= \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ -\triangle y + y &= 0 \quad \text{in } \Omega \\ \partial_n y &= u \quad \text{on } \Gamma \\ u_a \le u(x) &\le u_b \quad \text{a.e. on } \Gamma \\ y(x) &\ge y_c(x) \quad \text{a.e. in } \Omega', \end{aligned}\right\} \tag{P}$$

where Ω' is an inner subdomain that is strictly contained in Ω. The precise hypothesis on the given quantities in (P) are given in Assumption 1.1 below.

It is well known that problems with pointwise state constraints exhibits several difficulties caused by the low regularity of the Lagrange multipliers. We refer to Casas [4], where it is shown, that the Lagrange multipliers exist in general only in the space of regular Borel measures. This fact impairs the regularity of the optimal solution of (P) and consequently leads to numerical difficulties. In order to overcome this lack of regularity, different regularization concepts have been developed in the recent past, see for instance Ito and Kunisch [17], Hintermüller and Kunisch [14], Meyer, Rösch, and Tröltzsch [25], Meyer, Prüfert, and Tröltzsch [24], Cherednichenko and Rösch [6], and Tröltzsch and Yousept [30].

In this paper, we focus on a particular regularization approach, namely the concept of a *virtual distributed control* in the domain Ω that was introduced in [20]. Instead of problem (P), we will investigate a family of regularized optimal control problems:

$$\left.\begin{aligned} &\min \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ &\text{s. t.} \qquad -\triangle y + y = \phi(\varepsilon)v \qquad \text{in } \Omega \\ &\qquad\qquad \partial_n y = u \qquad \text{on } \Gamma \\ &\qquad\qquad u_a \le u(x) \le u_b \qquad \text{a.e. on } \Gamma \\ &\qquad\qquad y(x) \ge y_c(x) - \xi(\varepsilon)v(x) \qquad \text{a.e. in } \Omega', \end{aligned}\right\} \tag{P$^\varepsilon$}$$

with a regularization parameter $\varepsilon > 0$ and Ω, ν, y_d, y_c, u_a and u_b as defined above. The assumptions on the parameter functions are listed below in Assumption 1.2.

In the first part of this work, we lay the foundations for an a priori analysis of the *full finite element discretization* of problem (P$^\varepsilon$). To be more precise, we investigate the errors arising from the virtual control regularization and from the discretization of the boundary control in this part. The boundary control is discretized by linear and continuous ansatz functions. The second part [19] will finalize the overall analysis by incorporating the finite element discretization of the PDE into the discussion. A numerical validation of the theoretical results is also contained in the second part.

Because of the lack of regularity mentioned above, the a priori error analysis for state-constrained problems is known to be much more delicate compared to problems with pure control- or mixed control-state constraints. In the recent past, certain progress has been achieved concerning the finite element error analysis for linear-quadratic elliptic problems with distributed control. We refer for instance to [9],[10], and [23]. In the first two papers the so-called variational discretization concept introduced in [15] is transferred to the state-constrained case, whereas [23] deals with a full discretization. In addition, there

are several contributions concerning the discretization of regularized state-constrained problems with distributed control, see [22, 16, 13]. A more detailed overview over the existing literature for the numerical analysis of state-constrained problems will be given in the second part of this work.
All papers, mentioned above, deal with distributed controls and a priori error estimates for state-constrained problems with boundary control such as (P) have not been discussed so far. Thus, the consideration of boundary controls represents one of the genuine contributions of this paper. Under a suitable coupling of mesh size and regularization parameter ε, the final result of the first part reads as follows

$$\|\bar{u} - \bar{u}_\varepsilon^h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_\varepsilon^h\|_{L^2(\Omega)} \leq ch, \tag{1.1}$$

where h refers to the mesh size of the control discretization. The afore mentioned results for distributed controls indicate a significantly lower order of convergence for the case with boundary control than the one given in (1.1). The reason is that the state constraints are imposed in the inner subdomain Ω' which is strictly contained in Ω. This fact allows for a higher regularity of the optimal boundary control which is frequently used throughout the paper. It is another main novelty of the paper to exploit that the state constraints are only imposed in an inner subdomain of Ω. We point out that that such an assumption is fulfilled in many applications. As an instance we mention the sublimation growth of semiconductor single crystals by means of induction heating, see for e.g. [26]. Here, it is essential that the temperature inside the growth crucible does not exceed a certain threshold which mathematically corresponds to a state constraint in the interior as in (P). If in addition the PDE in (P^ε) is also discretized, the convergence order in (1.1) is nearly preserved as will be shown in the second part [19, Theorem 4.13].
The paper is organized as follows: In Section 2, we prove the additional regularity of the optimal solution that follows from the consideration of the state constraints in an inner subdomain. Section 3 is devoted to the discretization of the boundary control and the consideration of the corresponding semi-discretized and regularized optimal control problem. It also contains an important stability result for convex, discrete projections that is a generalization of a result in [5]. In Section 4, we construct feasible controls for the original problem (P) and the semi-discretized version of (P^ε). Based on these feasible controls, regularization and discretization error estimates, containing also (1.1), are derived in Section 5.

1.1. Assumptions and Notations. Let us briefly introduce the main notations used throughout the paper. If X is a Banach space, we denote its dual by X^*, and the associated dual pairing is $\langle \cdot\,, \cdot \rangle_{X,X^*}$. The space of regular Borel measures over a domain Ω is denoted by $\mathcal{M}(\Omega)$. If Ω_1 and Ω_2 are two open, bounded domains in $\mathbb{R}^d$, then we mean by $\Omega_1 \subset\subset \Omega_2$ that Ω_1 is strictly contained in Ω_2, i.e.,

$$\operatorname{dist}\{\overline{\Omega_1}, \partial\Omega_2\} := \inf_{x\in\overline{\Omega_1}, y\in\partial\Omega_2} \|x - y\|_{\mathbb{R}^d} > 0,$$

where $\|\,.\,\|_{\mathbb{R}^d}$ is the Euclidian norm. Next we state the basic assumptions, we require for the discussion of (P) and (P^ε), respectively:
ASSUMPTION 1.1. *The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is open, bounded, and convex with a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary Γ. Moreover, $\Omega' \subset\subset \Omega$ is an inner subdomain. Furthermore, $y_d \in L^2(\Omega)$, and $y_c \in C^{0,1}(\bar{\Omega})$ are given functions and $u_a \leq u_b$, $\nu > 0$ are real numbers.*
ASSUMPTION 1.2. *The functions ψ, ϕ and ξ are positive and real valued.*

2. Regularity and boundedness results. This section is concerned with regularity results for the state equation and first order optimality conditions for problem (P) and (P_ε), respectively. Furthermore, we derive several boundedness results for the optimal solution of the regularized problem.

2.1. Regularity for the state equation. We will start with the definition of a solution operator associated with the state equation. We introduce the following weak formulation of the state equation in problem (P) and (P_ε) for a right-hand side $f \in H^1(\Omega)^*$:

$$a(y, z) := \int_\Omega (\nabla y \cdot \nabla z + yz)dx = \langle f\,, z\rangle_{H^1(\Omega)^*,H^1(\Omega)}, \quad \forall z \in H^1(\Omega). \tag{2.1}$$

The Lax-Milgram Lemma gives the existence of a solution to (2.1) in $H^1(\Omega)$ for every element $f \in H^1(\Omega)^*$. The associated linear and continuous solution operator is denoted by $S : H^1(\Omega)^* \to H^1(\Omega)$. Next, we identify the right-hand sides of the state equations in (P) and (P$_\varepsilon$) with elements in $H^1(\Omega)^*$. Thanks to

$$\langle \tau^* u\, ,\, z\rangle_{H^1(\Omega)^*, H^1(\Omega)} := \int_\Gamma u \tau z \, ds, \tag{2.2}$$

where $\tau : H^1(\Omega) \to L^2(\Gamma)$ denotes the trace operator, the control $u \in L^2(\Gamma)$ defines an element in $H^1(\Omega)^*$. Furthermore, a virtual control $v \in L^2(\Omega)$ belongs to $H^1(\Omega)^*$ by

$$\langle E_H^* v\, ,\, z\rangle_{H^1(\Omega)^*, H^1(\Omega)} := \int_\Omega v E_H z \, dx, \tag{2.3}$$

where $E_H : H^1(\Omega) \to L^2(\Omega)$ is the embedding operator from $H^1(\Omega)$ to $L^2(\Omega)$. Hence, the weak solutions of the state equations of problem (P) and (P$_\varepsilon$) are given by:

$$y = S\tau^* u \quad \text{for (P)}, \qquad y_\varepsilon = S(\tau^* u^\varepsilon + \phi(\varepsilon) E_H^* v^\varepsilon) \quad \text{for (P}^\varepsilon\text{)}. \tag{2.4}$$

In the sequel, we will recall some regularity results for solutions of partial differential equations. Furthermore, we will discuss the smoothness of solutions in the interior of the domain Ω. This is essential here since the state constraints are only considered in an inner subdomain. We will start with a classical result of Grisvard, see [12].

THEOREM 2.1. *Let Ω be a convex, open, and polygonally or polyhedrally bounded domain in $\mathbb{R}^d$, $d = 2, 3$. Then, for every for every $(f, g) \in L^2(\Omega) \times H^{1/2}(\Gamma)$, the elliptic partial differential equation*

$$\begin{aligned} -\Delta w + w &= f \quad \text{in } \Omega \\ \partial_n w &= g \quad \text{on } \Gamma \end{aligned} \tag{2.5}$$

admits a unique solution $w \in H^2(\Omega)$, and there exists a constant $c > 0$ depending only on the domain such that

$$\|w\|_{H^2(\Omega)} \le c(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)})$$

is satisfied.

The next theorem is devoted to the higher interior regularity of weak solutions of elliptic partial differential equations. For the proof, we refer to [11, Chapter 6.3.1.].

THEOREM 2.2. *Let Ω be a convex, open and polygonally or polyhedrally bounded domain in $\mathbb{R}^d$, $d = 2, 3$. Suppose $w \in H^1(\Omega)$ is the weak solution of (2.5) for some $(f, g) \in L^2(\Omega) \times L^2(\Gamma)$. If additionally $f \in H^m(\Omega)$ for some nonnegative integer m, then w is an element of $H^{m+2}(U)$ for each subdomain $U \subset\subset \Omega$ and the estimate*

$$\|w\|_{H^{m+2}(U)} \le C(\|f\|_{H^m(\Omega)} + \|w\|_{L^2(\Omega)})$$

is satisfied, where the positive constant C is depending only on Ω, U and m.

This result already indicates that one benefits from the consideration of the state constraints in an inner subdomain of Ω. As an immediate consequence of Theorem 2.1 and Sobolev embeddings, we obtain the following corollary:

COROLLARY 2.3. *Let $y = S\tau^* u \in H^1(\Omega)$ for a given $u \in L^2(\Gamma)$. Furthermore, let $\Omega' \subset\subset \Omega$ be an inner subdomain of Ω. Then y is an element of $W^{2,\infty}(\Omega')$, and there exist a constant c, depending on Ω and Ω', such that*

$$\|y\|_{W^{2,\infty}(\Omega')} \le c\|y\|_{L^2(\Omega)}. \tag{2.6}$$

The $W^{2,\infty}$-regularity will also be essential for interior maximum norm estimates for finite element approximations to $y = S\tau^* u$ that are arising in the second part of this work, see [19]. In the previous corollary, the L^2-norm of the weak solution $y = S\tau^* u$ appears. The next lemma provides an estimate of this norm.

LEMMA 2.4. *Let $y = S\tau^* u \in H^1(\Omega)$ for a given $u \in L^2(\Gamma)$. Then there is a constant $c > 0$ independent of u such that*

$$\|y\|_{L^2(\Omega)} \leq c\,\|u\|_{H^1(\Gamma)^*}.$$

Proof. We introduce a dual problem for a given function $f \in L^2(\Omega)$:

$$a(z, w) = \int_\Omega f z\, dx, \quad \forall z \in H^1(\Omega),$$

where the bilinear form $a(\cdot, \cdot)$ is same defined as in (2.1). According to Theorem 2.1, there is a unique solution $w \in H^2(\Omega)$ and the estimate

$$\|w\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)} \tag{2.7}$$

is satisfied. Furthermore, we have

$$a(y, z) = \int_\Gamma u\tau z\, ds, \quad \forall z \in H^1(\Omega)$$

since $y \in H^1(\Omega)$ is the weak solution of the state equation of problem (P) for $u \in L^2(\Gamma)$. According to [27, Theorem II.4.11.], the trace operator τ is continuous from $H^2(\Omega)$ to $H^1(\Gamma)$. For a precise definition of $H^1(\Gamma)$ in case of polygonal or polyhedral boundaries, we refer to [27, Section II.4.3 p.88 ff.]. By means of the dual problem, estimate (2.7), and the continuity of the trace operator, we derive

$$\begin{aligned}
\|y\|_{L^2(\Omega)} &= \sup_{f \in L^2(\Omega)} \frac{|(f\,,\, y)_{L^2(\Omega)}|}{\|f\|_{L^2(\Omega)}} = \sup_{f \in L^2(\Omega)} \frac{|a(y, w)|}{\|f\|_{L^2(\Omega)}} \\
&= \sup_{f \in L^2(\Omega)} \frac{|(u\,,\, \tau w)_{L^2(\Gamma)}|}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f \in L^2(\Omega)} \frac{\|u\|_{H^1(\Gamma)^*}\|\tau w\|_{H^1(\Gamma)}}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f \in L^2(\Omega)} \frac{c\|u\|_{H^1(\Gamma)^*}\|w\|_{H^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \\
&\leq \sup_{f \in L^2(\Omega)} \frac{c\|u\|_{H^1(\Gamma)^*}\|f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \\
&= c\|u\|_{H^1(\Gamma)^*},
\end{aligned}$$

which is the assertion. □

2.2. Karush-Kuhn-Tucker conditions for (P)**.** In this section, we establish optimality conditions for problem (P) using a Lagrange multiplier approach for the state constraints in (P). Based on this, we will derive a certain smoothness properties of the optimal controls. First, we require the existence of an inner point w.r.t. the state constraints.

ASSUMPTION 2.5. *There exists a function $\hat{u} \in H^1(\Gamma)$ with $u_a \leq \hat{u}(x) \leq u_b$ a.e. on Γ and $\hat{y}(x) \geq y_c + \gamma$ a.e. in Ω' with $\gamma > 0$, where $\hat{y} = S\tau^*\hat{u}$.*

Due to this assumption, the admissible set of problem (P) is nonempty. Moreover, the set is convex and closed. Since the cost functional is strictly convex and radially unbounded, the existence and uniqueness of the optimal solution is obtained by standard methods. We point out that the existence of a feasible point is sufficient to derive this existence result. The stricter Assumption 2.5 is required to guarantee the existence of Lagrange multipliers associated with the state constraints.

It is well known that Lagrange multipliers associated with pointwise state constraints are in general only regular Borel measures. It is to be noted that the solution of the state equation is continuous in $\overline{\Omega'}$, see Theorem 2.2, such that Assumption 2.5 gives the existence of a Slater point with respect to the $C(\overline{\Omega'})$-topology. This allows to apply the generalized Karush-Kuhn-Tucker theory that implies the existence of a Lagrange multiplier. In the case of state-constrained optimal control problems, the theory was developed by Casas in [4]. Notice that the control constraints on the boundary are not treated by a Lagrange multiplier approach, and we define the following admissible set

$$U_{ad}^L := \{u \in L^2(\Gamma) : \ u_a \le u \le u_b \ \text{ a.e. on } \Gamma\}. \tag{2.8}$$

Adapting the theory of Casas in [4] to problem (P), we obtain the following result:

THEOREM 2.6. *Suppose that Assumption 2.5 is fullfilled. Moreover, let $(\bar{y}, \bar{u})$ be the optimal solution of problem* (P). *Then a regular Borel measure $\mu \in \mathcal{M}(\overline{\Omega'})$ and an adjoint state $p \in W^{1,s}(\Omega)$, $s < d/(d-1)$ exist such that the following optimality system is satisfied:*

$$\begin{aligned} -\Delta\bar{y} + \bar{y} &= 0 & \qquad -\Delta p + p &= \bar{y} - y_d - \chi_{\overline{\Omega'}}^* \mu \\ \partial_n \bar{y} &= \bar{u} & \partial_n p &= 0 \end{aligned} \tag{2.9}$$

$$(\tau p + \nu \bar{u}\,,\, u - \bar{u})_{L^2(\Gamma)} \ge 0, \quad \forall u \in U_{ad}^L \tag{2.10}$$

$$\begin{gathered} \int\limits_{\overline{\Omega'}} (y_c - \bar{y})\, d\mu = 0, \quad \bar{y}(x) \ge y_c(x) \quad \textit{for all } x \in \overline{\Omega'} \\ \int\limits_{\overline{\Omega'}} \varphi\, d\mu \ge 0 \quad \forall \varphi \in C(\overline{\Omega'})^+, \end{gathered} \tag{2.11}$$

where $\chi_{\overline{\Omega'}} : C(\Omega) \to C(\overline{\Omega'})$ denotes the restriction operator from Ω to $\overline{\Omega'}$. Moreover, $C(\overline{\Omega'})^+$ is defined by $C(\overline{\Omega'})^+ := \{y \in C(\overline{\Omega'}) \,|\, y(x) \ge 0\ \forall\, x \in \overline{\Omega'}\}$.

For a proof and a more detailed elaboration of this result, we refer to [4]. Here, a crucial problem in the case of boundary control problems with state constraints becomes visible: due to the structure of the variational inequality (2.10), the adjoint state is uniquely determined only on the boundary, cf. [1, Proposition 3.5]. Of course, the nonuniqueness of the dual variables causes severe problems for numerical optimization methods that tries to directly solve the full Karush-Kuhn-Tucker system (2.9)–(2.11).

Since the state constraints are only imposed in the inner subdomain Ω' and the Lagrange multiplier is only located there, one derives higher regularity of the adjoint state on the boundary Γ. This allows to increase the regularity of the optimal solution $(\bar{u}, \bar{y})$ of (P), which is demonstrated in the following. In view of the adjoint equation in Theorem 2.6, we consider the equation

$$\begin{aligned} -\Delta\, p + p &= \chi_{\overline{\Omega'}}^* \mu & \quad &\text{in } \Omega \\ \partial_n p &= 0 & &\text{on } \Gamma, \end{aligned} \tag{2.12}$$

with $\mu \in \mathcal{M}(\overline{\Omega'})$. Here, $\chi_{\overline{\Omega'}}^* : \mathcal{M}(\overline{\Omega'}) \to \mathcal{M}(\Omega)$ again denotes the adjoint of the restriction operator on $\overline{\Omega'}$. According to Casas [4], there is a unique solution of (2.12) in $W^{1,s}(\Omega)$, $s < d/(d-1)$, that fulfills

$$\|p\|_{W^{1,s}(\Omega)} \le c\, \|\chi_{\overline{\Omega'}}^* \mu\|_{\mathcal{M}(\Omega)} = c\, \|\mu\|_{\mathcal{M}(\overline{\Omega'})}. \tag{2.13}$$

However, on a domain that is separated from $\overline{\Omega'}$, p is more regular as stated in the following lemma.

LEMMA 2.7. *Let Ω'', and Ω''' be subdomains of Ω that satisfy*

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega''' \subset\subset \Omega.$$

Furthermore, let $p \in W^{1,s}(\Omega)$, $s < \frac{d}{d-1}$, be the solution of (2.12). *Then there is a constant $c > 0$ such that*

$$\|p\|_{H^2(\Omega \setminus \Omega''')} \le c\, \|\mu\|_{\mathcal{M}(\overline{\Omega'})},$$

where c only depends on Ω', Ω'', Ω''', and Ω.
Proof. We start by defining

$$\varphi \in C^\infty(\bar{\Omega}) \ , \ \varphi|_{\Omega'} \equiv 0 \ , \ \varphi|_{\Omega\setminus\Omega''} \equiv 1.$$

Note that such a function exists since $\operatorname{dist}(\partial\Omega'', \Omega') > 0$ by assumption. Then for every $z \in W^{1,s'}(\Omega)$, there holds

$$\begin{aligned}\int_\Omega (\nabla(p\varphi)\cdot\nabla z + (p\varphi)z)\,dx &= \int_\Omega (p\nabla\varphi\cdot\nabla z - z\nabla p\cdot\nabla\varphi)\,dx \\ &\quad + \underbrace{\int_\Omega (\nabla p\cdot\nabla(\varphi z) + p\,\varphi\,z)\,dx}_{=\int_{\overline{\Omega'}}\varphi\, z\, d\mu = 0} \\ &= -\int_\Omega (p\,z\,\Delta\varphi + 2\,z\nabla p\cdot\nabla\varphi)\,dx + \underbrace{\int_\Gamma z\,p\,\partial_n\varphi\,ds}_{=0},\end{aligned}$$

where we used $\nabla\varphi|_\Gamma = 0$, which holds due to $\operatorname{dist}(\partial\Omega, \Omega'') > 0$ and $\varphi|_{\Omega\setminus\Omega''} \equiv 1$. Hence we obtain the following variational formulation for $w := p\varphi$

$$\int_\Omega (\nabla w\cdot\nabla z + wz)\,dx = -\int_\Omega (p\,\Delta\varphi + 2\nabla p\cdot\nabla\varphi)z\,dx \quad \forall\, z \in W^{1,s'}(\Omega). \tag{2.14}$$

Clearly, due to the embedding $p \in W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$ and $\varphi \in C^\infty(\bar{\Omega})$, the right hand side in (2.14) defines an element of $H^1(\Omega)^*$ such that (2.14) admits a solution $w \in H^1(\Omega)$ giving in turn $p \in H^1(\Omega \setminus \Omega'')$ by the definition of φ. Next we repeat the argument w.r.t. Ω''', i.e., we define a function ψ with

$$\psi \in C^\infty(\bar{\Omega}) \ , \ \psi|_{\Omega''} \equiv 0 \ , \ \psi|_{\Omega\setminus\Omega'''} \equiv 1.$$

Then $\zeta := w\psi$ solves for all $z \in H^1(\Omega)$

$$\begin{aligned}\int_\Omega (\nabla\zeta\cdot\nabla z + \zeta z)\,dx &= -\int_\Omega (w\,\Delta\psi + 2\nabla w\cdot\nabla\psi)z\,dx + \int_\Omega (\nabla w\cdot\nabla(\psi z) + w\,\psi\,z)\,dx \\ &= -\int_\Omega \big[w\,\Delta\psi + 2\nabla w\cdot\nabla\psi + \chi_{\Omega''}(p\,\Delta\varphi + 2\nabla p\cdot\nabla\varphi)\psi\big]\,z\,dx,\end{aligned}$$

where $\chi_{\Omega''}$ denotes the characteristic function on Ω''. Notice that we used (2.14) and $\psi|_{\Omega''} = 0$ for the last equality. Due to $p \in H^1(\Omega \setminus \Omega'')$, $w \in H^1(\Omega)$, and $\varphi, \psi \in C^\infty(\bar{\Omega})$ we have

$$w\,\Delta\psi + 2\nabla w\cdot\nabla\psi + \chi_{\Omega''}(p\,\Delta\varphi + 2\nabla p\cdot\nabla\varphi)\psi \in L^2(\Omega),$$

and consequently $\zeta \in H^2(\Omega)$ by Theorem 2.1, implying in turn $p \in H^2(\Omega \setminus \Omega''')$. The estimate on $\|p\|_{H^2(\Omega\setminus\Omega''')}$ finally follows by straight forward estimation from (2.13) and the estimate in Theorem 2.1. ∎

Based on this lemma and the trace theorem in [27, Theorem II.4.11.], we infer:
Corollary 2.8. *Let the assumptions of Theorem 2.6 be fulfilled. Moreover, let $\mu \in \mathcal{M}(\overline{\Omega'})$ be a regular Borel measure and $p \in W^{1,s}(\Omega)$, $s < d/(d-1)$, an adjoint state such that the optimality system (2.9)-(2.11) is satisfied. Then, we have $p \in H^1(\Gamma)$ and there is a positive constant $c > 0$ such that*

$$\|p\|_{H^1(\Gamma)} \le c(\|\bar{y}\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)} + \|\mu\|_{\mathcal{M}(\overline{\Omega'})}) \tag{2.15}$$

is valid.

Next, we introduce the projection operator $P : L^2(\Gamma) \to L^2(\Gamma)$ on the admissible set U_{ad}^L, given by

$$P(\bar{w}) := \arg \min_{w \in U_{ad}^L} \frac{1}{2} \|w - \bar{w}\|_{L^2(\Gamma)}^2 \tag{2.16}$$

for given $\bar{w} \in L^2(\Gamma)$. By standard arguments, one shows that $P(\bar{w})$ is the unique solution of

$$(P(\bar{w}) - \bar{w}\, ,\, w - P(\bar{w}))_{L^2(\Gamma)} \geq 0 \qquad \forall w \in U_{ad}^L. \tag{2.17}$$

Hence, the variational inequality (2.10) is equivalent to

$$\bar{u} = P\left\{-\frac{\tau p}{\nu}\right\}. \tag{2.18}$$

Moreover, a pointwise evaluation of (2.17) implies

$$P(\bar{w})(x) = \max\{u_a, \min\{\bar{w}(x), u_b\}\} \quad \text{a.e. on } \Gamma,$$

i.e., P is equivalent to the pointwise projection on U_{ad}^L.

Lemma 2.9. *Let $\bar{w} \in H^1(\Gamma)$ be a given function. Then, we have $P(\bar{w}) \in H^1(\Gamma)$ and there exist positive constants C_1 and C_2, depending on the boundary and the bounds u_a, u_b, such that*

$$\|P(\bar{w})\|_{H^1(\Gamma)} \leq C_1 \|\bar{w}\|_{H^1(\Gamma)} + C_2$$

holds true.

For the corresponding proof, we refer to [18] and [21]. Thanks to Lemma 2.9 and Corollary 2.8, the optimal control $\bar{u}$ belongs to $H^1(\Gamma)$ and there exists a constant $C > 0$ such that

$$\|\bar{u}\|_{H^1(\Gamma)} \leq C. \tag{2.19}$$

is satisfied. The higher regularity of the optimal control improves the regularity of the associated optimal state. According to Theorem 2.1, the optimal state $\bar{y} = S\tau^*\bar{u}$ belongs to $H^2(\Omega)$ and the estimate

$$\|\bar{y}\|_{H^2(\Omega)} \leq c\|\bar{u}\|_{H^{1/2}(\Gamma)}$$

is satisfied for constant depending only on the domain.

2.3. The regularized problem (P$^\varepsilon$)**.** In the sequel, we introduce the optimality conditions of problem (P$^\varepsilon$) using a Lagrange multiplier approach for the mixed control-state constraints. Similarly to the unregularized problem, the existence and uniqueness of an optimal solution ($\bar{y}_\varepsilon$, $\bar{u}_\varepsilon$, $\bar{v}_\varepsilon$) for problem (P$^\varepsilon$) is obtained by standard arguments, if Assumption 2.5 is satisfied, since the control $(\hat{u}, 0) \in L^2(\Gamma) \times L^2(\Omega)$ is feasible for problem (P$^\varepsilon$).

Similarly to above, the control constraints in (P$^\varepsilon$) are treated by the admissible set U_{ad}^L defined in (2.8). We point out that in the case of pointwise control-state-constraints the Lagrange multipliers are regular functions, see e.g. [2], [28], or [29]. By applying analysis of [28], one obtains the following first-order optimality conditions for (P$^\varepsilon$):

Proposition 2.10. *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon) \in H^1(\Omega) \times L^2(\Gamma) \times L^2(\Omega)$ be the optimal solution of* (P$^\varepsilon$)*. Then, there exist a unique adjoint state $p_\varepsilon \in H^1(\Omega)$ and a unique Lagrange multiplier $\mu_\varepsilon \in L^2(\Omega')$ so that the following optimality system is satisfied*

$$\begin{aligned} -\triangle \bar{y}_\varepsilon + \bar{y}_\varepsilon &= \phi(\varepsilon)\bar{v}_\varepsilon & \qquad -\triangle p_\varepsilon + p_\varepsilon &= \bar{y}_\varepsilon - y_d - E_{\Omega'}^* \mu_\varepsilon \\ \partial_n \bar{y}_\varepsilon &= \bar{u}_\varepsilon & \partial_n p &= 0 \end{aligned} \tag{2.20}$$

$$(\tau p_\varepsilon + \nu \bar{u}_\varepsilon\, ,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{ad}^L \tag{2.21}$$

$$\phi(\varepsilon) p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon) E_{\Omega'}^* \mu_\varepsilon = 0 \quad \textit{a.e. in } \Omega \tag{2.22}$$

$$(\mu_\varepsilon\, ,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} = 0, \quad \mu_\varepsilon \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \textit{a.e. in } \Omega', \tag{2.23}$$

where $E_{\Omega'} : L^2(\Omega) \to L^2(\Omega')$ denotes the respective restriction operator to Ω'.

REMARK 2.11. *It is to be noted that the Lagrange multiplier as well as the adjoint state for* (P^ε) *are unique in contrast to the case with pure state constraints, see Theorem 2.6. This is one of the major advantages of the regularization which is especially important for numerical algorithms that rely on the use of dual variables.*

Notice that, for fixed $\varepsilon > 0$, the regularity of μ_ε and $\bar{p}_\varepsilon$ can even be increased. However, for the subsequent limit analysis for ε tending to zero, uniform boundedness of μ_ε and $\bar{p}_\varepsilon$ w.r.t. ε is required. The next lemma shows, that the multiplier μ_ε is uniformly bounded in $L^1(\Omega')$ for every $\varepsilon > 0$. The proof follows a strategy analogous to [22, Lemma 2.2].

LEMMA 2.12. *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be the optimal solution of problem* (P^ε)*. Furthermore, let p_ε be the adjoint state and μ_ε the Lagrange multiplier, such that the optimality system* (2.20)-(2.23) *is fulfilled. Then, the Lagrange multiplier μ_ε is uniformly bounded in $L^1(\Omega')$, i.e.*

$$\|\mu_\varepsilon\|_{L^1(\Omega')} \le C, \tag{2.24}$$

where the constant $C > 0$ is independent of the regularization parameter ε.

Proof. First, we rewrite the equation (2.22) in a variational form

$$(\phi(\varepsilon) E_H p_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon) E^*_{\Omega'}\mu_\varepsilon \,,\, v - \bar{v}_\varepsilon)_{L^2(\Omega)} = 0 \quad \forall v \in L^2(\Omega).$$

Adding the previous variational equation and (2.21) and using the representation of the adjoint state p_ε by the adjoint of the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$, we arrive at

$$\begin{aligned}(E^*_{\Omega'}\mu_\varepsilon \,,\, \xi(\varepsilon)(v - \bar{v}_\varepsilon) + S E^*_H \phi(\varepsilon)(v - \bar{v}_\varepsilon) + S\tau^*(u - \bar{u}_\varepsilon))_{L^2(\Omega)}& \\ \le (\psi(\varepsilon)\bar{v}_\varepsilon + \phi(\varepsilon) E_H S^*(\bar{y}_\varepsilon - y_d) \,,\, v - \bar{v}_\varepsilon)_{L^2(\Omega)}& \\ + (\nu \bar{u}_\varepsilon + \tau S^*(\bar{y}_\varepsilon - y_d) \,,\, u - \bar{u}_\varepsilon)_{L^2(\Gamma)}&,\end{aligned} \tag{2.25}$$

for all $(u, v) \in U^L_{ad} \times L^2(\Omega)$. Now, we choose the special test function $(\hat{u}, 0) \in U^L_{ad} \times L^2(\Omega)$, where $\hat{u}$ is the inner point with respect to the pure state constraints defined in Assumption 2.5. Using (2.4), we find for the left hand side of the previous inequality (2.25)

$$\begin{aligned}(E^*_{\Omega'}\mu_\varepsilon \,,\, \xi(\varepsilon)(-\bar{v}_\varepsilon) + S E^*_H \phi(\varepsilon)(-\bar{v}_\varepsilon) + S\tau^*(\hat{u} - \bar{u}_\varepsilon))_{L^2(\Omega)}& \\ = (\mu_\varepsilon \,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega')} + (E^*_{\Omega'}\mu_\varepsilon \,,\, \hat{y} - y_c)_{L^2(\Omega)}& \\ = (E^*_{\Omega'}\mu_\varepsilon \,,\, \hat{y} - y_c)_{L^2(\Omega)}&,\end{aligned} \tag{2.26}$$

since the first term in the second line vanishes by (2.23). With the help of Assumption 2.5 and the positivity of the Lagrange multiplier, one derives the estimate

$$\gamma \|\mu_\varepsilon\|_{L^1(\Omega')} = \int_{\Omega'} \gamma \mu_\varepsilon dx \le (E^*_{\Omega'}\mu_\varepsilon \,,\, \hat{y} - y_c)_{L^2(\Omega)}. \tag{2.27}$$

We note that $E^*_{\Omega'} : L^2(\Omega') \to L^2(\Omega)$ represents the extension by zero on $\Omega \setminus \Omega'$. Summarizing (2.25) for $(\hat{u}, 0) \in U^L_{ad} \times L^2(\Omega)$, (2.26) and (2.27), we conclude

$$\begin{aligned}\gamma \|\mu_\varepsilon\|_{L^1(\Omega')} \le (E^*_{\Omega'}\mu_\varepsilon \,,\, \hat{y} - y_c)_{L^2(\Omega)} \le (\psi(\varepsilon)\bar{v}_\varepsilon + \phi(\varepsilon) E_H S^*(\bar{y}_\varepsilon - y_d) \,,\, -\bar{v}_\varepsilon)_{L^2(\Omega)}& \\ + (\nu \bar{u}_\varepsilon + \tau S^*(\bar{y}_\varepsilon - y_d) \,,\, \hat{u} - \bar{u}_\varepsilon)_{L^2(\Gamma)}&.\end{aligned}$$

This implies

$$\begin{aligned}\gamma \|\mu_\varepsilon\|_{L^1(\Omega')} \le& - \psi(\varepsilon)\|\bar{v}_\varepsilon\|^2_{L^2(\Omega)} + (\bar{y}_\varepsilon - y_d \,,\, -S E^*_H \phi(\varepsilon)\bar{v}_\varepsilon)_{L^2(\Omega)} \\ &+ (\bar{y}_\varepsilon - y_d \,,\, S\tau^*(\hat{u} - \bar{u}_\varepsilon))_{L^2(\Omega)} + \nu(\bar{u}_\varepsilon \,,\, \hat{u} - \bar{u}_\varepsilon)_{L^2(\Gamma)} \\ \le& (y_d \,,\, \bar{y}_\varepsilon)_{L^2(\Omega)} + \nu(\bar{u}_\varepsilon \,,\, \hat{u})_{L^2(\Gamma)} + (\bar{y}_\varepsilon - y_d \,,\, \hat{y})_{L^2(\Omega)} \\ \le& \|y_d\|_{L^2(\Omega)}\|\bar{y}_\varepsilon\|_{L^2(\Omega)} + \nu\|\bar{u}_\varepsilon\|_{L^2(\Gamma)}\|\hat{u}\|_{L^2(\Gamma)} \\ &+ \|\bar{y}_\varepsilon - y_d\|_{L^2(\Omega)}\|\hat{y}\|_{L^2(\Omega)},\end{aligned}$$

where we again used (2.4). The optimality of $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ yields the uniform boundedness of the remaining terms in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively, independently of ε This completes the proof. ∎

COROLLARY 2.13. *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfy the optimality system (2.20)-(2.23) with associated adjoint state p_ε and Lagrange multiplier μ_ε. Then, there exists a constant $C > 0$ independent of ε such that*

$$\|p_\varepsilon\|_{H^1(\Gamma)} \leq C. \tag{2.28}$$

is satisfied.

Proof. The arguments are similar to Corollary 2.8. The standard result of Grisvard, see Theorem 2.1, and the trace theorem [27, Theorem II.4.11.] provides the estimate for the adjoint state with respect to the regular part by $\bar{y}_\varepsilon$ and y_d. The assertion then follows from Lemma 2.7 and 2.12. ∎

Analogous to the original problem (P) in the previous section, the variational inequality (2.10) can be replaced

$$\bar{u}_\varepsilon = P\left\{-\frac{\tau p_\varepsilon}{\nu}\right\},$$

where P again denotes the projection on the admissible set U_{ad}^L. By the means of Lemma 2.9 and Corollary 2.13, we obtain the boundedness of the regularized optimal control in $H^1(\Gamma)$, i.e.,

$$\|\bar{u}_\varepsilon\|_{H^1(\Gamma)} \leq C \tag{2.29}$$

for some constant C independent of ε.

3. Semi-discretization. One of the main difficulties in deriving discretization error estimates for optimal control problems is caused by the discretization of the control. Thus, we will focus on the discretization of the boundary control here, while the discretization of the virtual control and the state equation is postponed to the second part [19].

3.1. Discretization of the boundary control. We start with a mesh $\mathcal{T}_h$ of pairwise disjoint open elements T with

$$\Gamma = \bigcup_{i=1}^{n_\Gamma} \bar{T}.$$

Note that Γ is a polygon or polyhedron such that the meshes are easily constructed, and they exactly fit the boundary. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $R(T)$, where $\rho(T)$ denotes the diameter of the set T and $R(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We suppose the following regularity assumption for $\mathcal{T}_h$:

ASSUMPTION 3.1. *There exist two positive constants ρ and R such that*

$$\frac{\rho(T)}{R(T)} \leq R, \quad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

The number of elements of the mesh $\mathcal{T}_h$ is denoted by n_Γ. Furthermore, the vertices of the elements in $\mathcal{T}_h$ are denoted by x_i, $i = 1, \ldots, n_e$.

Based on this mesh, the space of discrete boundary controls is defined by

$$U_h = \{u \in C(\Gamma) \,|\, v|_{T_j} \in \mathcal{P}_1 \text{ for } j = 1, \ldots, n_\Gamma\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal 1.

DEFINITION 3.2. *As basis for the finite dimensional space U_h we choose the functions $\varphi_i \in U_h$, $i = 1, ..., n_e$, that satisfy $\varphi_i(x_j) = \delta_{ij}$. Note that these functions fulfill*

$$\varphi_i(x) \geq 0 \text{ a.e. on } \Gamma, \qquad \sum_{i=1}^{n_e} \varphi_i(x) = 1. \tag{3.1}$$

Remark 3.3. *We define by*

$$\omega_i := supp\ \varphi_i \qquad i = 1, \ldots, n_e$$

the patch ω_i that consists of the M_i adjacent elements of $\mathcal{T}_h$ that share the vertex x_i. Assumption 3.1 implies that there exists a constant $M \in \mathbb{N}$, independent of h, such that $M_i \leq M$ for all $i = 1, \ldots, n_e$.
Now, we define a quasi-interpolation operator as introduced in [3]. For an arbitrary $u \in L^1(\Gamma)$, the operator is constructed as follows:

$$\Pi_h u = \sum_{i=1}^{n_e} \pi_i(u)\varphi_i, \tag{3.2}$$

where the coefficients $\pi_i(u) \in \mathbb{R}$ are defined by

$$\pi_i(u) = \frac{\int_{\omega_i} u\varphi_i ds}{\int_{\omega_i} \varphi_i ds}. \tag{3.3}$$

It is easily seen that Π_h satisfies

$$u_a \leq u(x) \leq u_b \text{ a.e. on } \Gamma \quad \Rightarrow \quad u_a \leq (\Pi_h u)(x) \leq u_b \text{ a.e. on } \Gamma, \tag{3.4}$$

i.e., Π_h preserves feasibility w.r.t. the control constraints. We point out that this property is essential for the subsequent analysis. Note that the standard L^2-projection as well as the classical quasi-interpolation, introduced in [7], does not have the property (3.4).
Forthcoming, we will state error estimates for $u - \Pi_h u$ in different norms. The underlying analysis was developed in [8] for functions defined in the domain. However, the proof can be easily adapted to the boundary case, so we skip the proof.
Lemma 3.4. *There is a constant c, independent of h, such that*

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq ch\|u\|_{H^1(\Gamma)} \tag{3.5}$$
$$\|u - \Pi_h u\|_{H^1(\Gamma)^*} \leq ch^2\|u\|_{H^1(\Gamma)} \tag{3.6}$$

for all $u \in H^1(\Gamma)$.
The proof is along the lines of the results given in [8, Lemma 4.4 and 4.5] for functions defined in the domain. Moreover, we will need the following result:
Lemma 3.5. *The quasi-interpolation operator is stable w.r.t. the L^2-norm, i.e., for every $u \in L^2(\Gamma)$, there holds*

$$\|\Pi_h u\|_{L^2(\Gamma)} \leq c\,\|u\|_{L^2(\Gamma)}$$

with a constant $c > 0$ independent of h.
Proof. By Assumption 3.1, there is a number $N \in \mathbb{N}$, independent of h, such that $\max_{i\in\{1,\ldots,n_e\}} |\{k \in \{1, ..., n_e\} : \omega_j \cap \omega_k \neq \emptyset\}| = N < d\,M$ with M as defined in Remark 3.3. The assertion then easily follows from (3.2) and (3.3):

$$\begin{aligned}
\|\Pi_h\, u\|^2_{L^2(\Gamma)} &= \int_\Gamma \Big[\sum_{i=1}^{n_e} \pi_i(u)^2\varphi_i^2 + 2\sum_{j=1}^{n_e-1}\sum_{k=j+1}^{n_e} \pi_j(u)\varphi_j\,\pi_k(u)\varphi_k\Big]\,ds \\
&\leq \int_\Gamma \Big[\sum_{i=1}^{n_e} \pi_i(u)^2\varphi_i^2 + \sum_{j=1}^{n_e-1}\sum_{\substack{\omega_j\cap\omega_k\neq\emptyset \\ k>j}} \big(\pi_j(u)^2\varphi_j^2 + \pi_k(u)^2\varphi_k^2\big)\Big]\,ds \\
&\leq (N+1)\sum_{i=1}^{n_e}\int_\Gamma \pi_i(u)^2\varphi_i^2\,ds = (N+1)\sum_{i=1}^{n_e}\int_{\omega_i} u^2\,ds\left(\frac{\int_{\omega_i}\varphi_i^2\,ds}{\int_{\omega_i}\varphi_i\,ds}\right)^2 \leq c\,\|u\|_{L^2(\Gamma)},
\end{aligned}$$

where we used that $0 \leq \varphi_i(x) \leq 1$ for all $x \in \Gamma$ and all $i \in \{1, ..., n_e\}$. $\square$

The stability of Π_h in the H_0^1-seminorm is shown in [3, Theorem 3.1] for functions defined in the domain. The arguments are based on the fact that $\sum_i \nabla\varphi_i(x) = 0$ and can easily be adapted to the boundary case so that, together with Lemma 3.5,

$$\|\Pi_h u\|_{H^1(\Gamma)} \leq c\,\|u\|_{H^1(\Gamma)} \quad \forall\, u \in H^1(\Gamma) \tag{3.7}$$

is obtained. This inequality will be useful in the upcoming analysis.

3.2. The Semi-discretized and regularized optimal control problem. Associated to the finite element space U_h for the boundary control, introduced in the previous section, we consider the following semi-discretized and regularized problem:

$$\left.\begin{aligned} &\min \quad J(y_\varepsilon^h, u_\varepsilon^h, v_\varepsilon^h) = \frac{1}{2}\|y_\varepsilon^h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_\varepsilon^h\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon^h\|^2_{L^2(\Omega)} \\ &\text{s.t.} \qquad\quad y_\varepsilon^h = S(\tau^* u_\varepsilon^h + \phi(\varepsilon)E_H^* v_\varepsilon^h) \quad \text{and} \quad (u_\varepsilon^h, v_\varepsilon^h) \in V_{ad}^{\varepsilon,h}, \end{aligned}\right\} \tag{P_h^ε}$$

where the admissible set is defined by

$$\begin{aligned} V_{ad}^{\varepsilon,h} := \{&(u_\varepsilon^h, v_\varepsilon^h) \in U_h \times L^2(\Omega) \,|\, u_a \leq u_\varepsilon^h(x) \leq u_b \text{ a.e. on } \Gamma, \\ &S(\tau^* u_\varepsilon^h + \phi(\varepsilon)E_H^* v_\varepsilon^h)(x) \geq y_c(x) - \xi(\varepsilon)v_\varepsilon^h(x) \text{ a.e. in } \Omega'\}. \end{aligned}$$

The admissible set is convex and closed. Based on Assumption 2.5, the next lemma shows that the admissible set is nonempty for sufficiently small h. Thus, the problem (P_h^ε) admits a unique solution.

LEMMA 3.6. *There is an $h_0 > 0$ such that, for all $h \leq h_0$*

$$\hat{y}^h(x) = (S\tau^*\Pi_h\hat{u})(x) \geq y_c(x) + \gamma_0, \qquad \text{a.e. in } \Omega'$$

is valid with a constant γ_0 independent of h.

Proof. Since $\hat{u}$ satisfies the control constraints and the quasi-interpolation operator Π_h by (3.2) preserves this property, we obtain $u_a \leq \Pi_h\hat{u} \leq u_b$. With the help of Assumption 2.5, we proceed with

$$\begin{aligned} (S\tau^*\Pi_h\hat{u})(x) &= (S\tau^*\hat{u})(x) + (S\tau^*(\Pi_h\hat{u} - \hat{u}))(x) \\ &\geq y_c(x) + \gamma - \|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^\infty(\Omega')}. \end{aligned}$$

The L^∞-estimate in the subdomain Ω' is estimated by Corollary 2.3 and Lemma 2.4, which gives

$$\|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^\infty(\Omega')} \leq c\|S\tau^*(\Pi_h\hat{u} - \hat{u})\|_{L^2(\Omega)} \leq c\|\Pi_h\hat{u} - \hat{u}\|_{H^1(\Gamma)^*}.$$

Thanks to (3.6), we end up with

$$\hat{y}^h(x) \geq y_c(x) + \gamma - ch^2\|\hat{u}\|_{H^1(\Gamma)}.$$

Hence, if $h = h_0$ is chosen sufficiently small, we obtain the assertion with $\gamma_0 := \gamma - ch_0^2\|\hat{u}\|_{H^1(\Gamma)} > 0$. ☐

Next we establish first-order optimality conditions for problem (P_h^ε) using a Lagrange multiplier approach for the mixed constraints. The associated Lagrange multiplier is denoted by μ_ε^h. As in case of (P^ε), it is a proper function for every $\varepsilon > 0$. The control constraints are still treated by an admissible set:

$$U_{h,ad}^L := \{u \in U_h : \quad u_a \leq u \leq u_b \text{ a.e. on } \Gamma\}.$$

By standard arguments, one derives the following result which is the analog to Proposition 2.10:

PROPOSITION 3.7. *Suppose that $(\bar{y}_\varepsilon^h, \bar{u}_\varepsilon^h, \bar{v}_\varepsilon^h) \in H^1(\Omega) \times U_h \times L^2(\Omega)$ is the unique solution of (P_h^ε). Then a unique adjoint state $\bar{p}_\varepsilon^h \in H^1(\Omega)$ and a unique Lagrange multiplier $\mu_\varepsilon^h \in L^2(\Omega)$ exist such that the optimality system, given by*

$$\bar{y}_\varepsilon^h = S(\tau^*\bar{u}_\varepsilon^h + \phi(\varepsilon)E_H^*\bar{v}_\varepsilon^h) \tag{3.8}$$

$$p_\varepsilon^h = S^*(\bar{y}_\varepsilon^h - y_d - E_{\Omega'}'\mu_\varepsilon^h) \tag{3.9}$$

$$\left(\tau p_\varepsilon^h + \nu\bar{u}_\varepsilon^h, u - \bar{u}_\varepsilon^h\right)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U_{h,ad}^L \tag{3.10}$$

$$\phi(\varepsilon)p_\varepsilon^h + \psi(\varepsilon)\bar{v}_\varepsilon^h - \xi(\varepsilon)E_{\Omega'}'\mu_\varepsilon^h = 0 \quad \text{a.e. in } \Omega \tag{3.11}$$

$$\begin{gathered} (\mu_\varepsilon^h\,,\, y_c - \bar{y}_\varepsilon^h - \xi(\varepsilon)\bar{v}_\varepsilon^h)_{L^2(\Omega')} = 0, \\ \mu_\varepsilon^h \geq 0, \quad \bar{y}_\varepsilon^h \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon^h \quad \text{a.e. in } \Omega', \end{gathered} \tag{3.12}$$

is fulfilled.
Similar to the continuous problem in Section 2.3, one derives a uniform bound of the multiplier μ_ε^h.
LEMMA 3.8. *Let* $(\bar{y}_\varepsilon^h, \bar{u}_\varepsilon^h, \bar{v}_\varepsilon^h)$ *be the optimal solution of problem* $(\mathrm{P}_h^\varepsilon)$. *Furthermore, let* p_ε^h *be the adjoint state and* μ_ε^h *the Lagrange multiplier, such that the optimality system* (3.8)-(3.12) *is fulfilled. Then for all* $h \le h_0$, *the Lagrange multiplier* μ_ε^h *is uniformly bounded in* $L^1(\Omega')$, *i.e.*

$$\|\mu_\varepsilon^h\|_{L^1(\Omega')} \le C, \tag{3.13}$$

where the constant $C > 0$ *is independent of the regularization parameter* ε *and the mesh size* h.
The proof is along the lines of the proof of Lemma 2.12 using the optimal solution of $(\mathrm{P}_h^\varepsilon)$ and Lemma 3.6. Furthermore, this result yields the uniform boundedness of the adjoint state p_ε^h, similarly to Corollary 2.8.
COROLLARY 3.9. *Let* $(\bar{y}_\varepsilon^h, \bar{u}_\varepsilon^h, \bar{v}_\varepsilon^h)$ *satisfy the optimality system (2.20)-(2.23) with associated adjoint state* p_ε^h *and Lagrange multiplier* μ_ε^h. *Then, there is a constant* $C > 0$, *independent of* ε *and* h, *such that*

$$\|p_\varepsilon^h\|_{H^1(\Gamma)} \le C. \tag{3.14}$$

holds true.

3.3. Boundedness of the discrete control - stability of convex projections. This section is devoted to the uniform boundedness of the discrete optimal control $\bar{u}_\varepsilon^h$ in $H^1(\Gamma)$. To this end, we investigate the projections on the convex sets U_{ad}^L and $U_{h,ad}^L$, respectively. Recall that the L^2-projection on U_{ad}^L, defined in (2.16), satisfies the variational inequality (2.17). Analogously, the discrete counterpart $P_h(\bar{w})$ for a given $\bar{w} \in L^2(\Gamma)$ denotes the solution of

$$\min_{w_h \in U_{h,ad}^L} \frac{1}{2}\|w_h - \bar{w}\|_{L^2(\Gamma)}^2,$$

which is equivalent to

$$(P_h(\bar{w}) - \bar{w}\,,\, w_h - P_h(\bar{w}))_{L^2(\Gamma)} \ge 0 \qquad \forall w_h \in U_{h,ad}^L. \tag{3.15}$$

LEMMA 3.10. *Let* $\bar{w} \in H^1(\Gamma)$ *be given. Furthermore, let* $P(\bar{w})$ *be the solution of* (2.17), *while* $P_h(\bar{w})$ *is the solution of* (3.15), *respectively. Then, there exists a positive constant* c, *independent of* h, *such that*

$$\|P_h(\bar{w}) - P(\bar{w})\|_{L^2(\Gamma)} \le ch\|\bar{w}\|_{H^1(\Gamma)} \tag{3.16}$$

is valid.
Proof. We start with the variational inequalities (2.17) and (3.15), respectively. Clearly, $P_h(\bar{w})$ is feasible for (2.17) such that

$$(P(\bar{w}) - \bar{w}\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} \ge 0. \tag{3.17}$$

Since $P(\bar{w})$ is the solution of the variational inequality (2.17) and the operator Π_h, defined in (3.2), preserves the validity of the inequality constraints, we obtain $\Pi_h(P(\bar{w})) \in U_{h,ad}^L$. Thus, we are allowed to choose $\Pi_h(P(\bar{w}))$ as a feasible function in (3.15). Adding the arising inequality and (3.17) yields

$$\begin{aligned}
0 &\le (P(\bar{w}) - \bar{w}\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P_h(\bar{w}))_{L^2(\Gamma)} \\
0 &\le (P(\bar{w}) - P_h(\bar{w})\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} + (P_h(\bar{w}) - \bar{w}\,,\, P_h(\bar{w}) - P(\bar{w}))_{L^2(\Gamma)} \\
&\quad + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P_h(\bar{w}))_{L^2(\Gamma)} \\
0 &\le -\|P_h(\bar{w}) - P(\bar{w})\|_{L^2(\Gamma)}^2 + (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)}.
\end{aligned}$$

We continue with

$$\begin{aligned}
\|P_h(\bar{w}) - P(\bar{w})\|_{L^2(\Gamma)}^2 &\le (P_h(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)} \\
&= (P_h(\bar{w}) - P(\bar{w})\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)} \\
&\quad + (P(\bar{w}) - \bar{w}\,,\, \Pi_h(P(\bar{w})) - P(\bar{w}))_{L^2(\Gamma)}.
\end{aligned}$$

Applying Young's inequality to the first term, we obtain

$$\begin{aligned}\frac{1}{2}\|P_h(\bar w)-P(\bar w)\|^2_{L^2(\Gamma)} \le \frac{1}{2}&\|\Pi_h(P(\bar w))-P(\bar w)\|^2_{L^2(\Gamma)}\\ &+\|P(\bar w)-\bar w\|_{H^1(\Gamma)}\|\Pi_h(P(\bar w))-P(\bar w)\|_{(H^1(\Gamma))^*}.\end{aligned}$$

With the help of the approximation error estimates (3.5) and (3.6), we obtain

$$\frac{1}{2}\|P_h(\bar w)-P(\bar w)\|^2_{L^2(\Gamma)} \le ch^2\|P(\bar w)\|^2_{H^1(\Gamma)} + ch^2\|P(\bar w)-\bar w\|_{H^1(\Gamma)}\|P(\bar w)\|_{H^1(\Gamma)}.$$

As stated in Lemma 2.9, P is stable in $H^1(\Gamma)$ such that

$$\|P_h(\bar w)-P(\bar w)\|_{L^2(\Gamma)} \le ch\|\bar w\|_{H^1(\Gamma)}$$

with a positive constant c independent of the mesh size h. □

Now, we can state the uniform boundedness of the discrete optimal control $\bar u^h_\varepsilon$ in $H^1(\Gamma)$ w.r.t. ε and h.

LEMMA 3.11. *Let $\bar u^h_\varepsilon \in U^L_{h,ad}$ be the discrete optimal control determined by the optimality system* (3.8)-(3.12). *Then there exists a positive constant C independent of h and ε such that*

$$\|\bar u^h_\varepsilon\|_{H^1(\Gamma)} \le C$$

is satisfied.

Proof. The variational inequality (3.10) can be interpreted as the projection of $-\bar p^h_\varepsilon/\nu$ on the convex set $U^L_{h,ad}$, e.g.

$$\bar u^h_\varepsilon = P_h(-p^h_\varepsilon/\nu).$$

Introducing the projection $P(-p^h_\varepsilon/\nu)$ according to the variational inequality (2.17) and applying the triangle inequality, we obtain

$$\|\bar u^h_\varepsilon\|_{H^1(\Gamma)} \le \|P_h(-p^h_\varepsilon/\nu)-\Pi_h(P(-p^h_\varepsilon/\nu))\|_{H^1(\Gamma)} + \|\Pi_h(P(-p^h_\varepsilon/\nu))\|_{H^1(\Gamma)}, \tag{3.18}$$

with the quasi-interpolation operator Π_h, defined in (3.2). Thanks to Corollary 3.9, Lemma 2.9, and Lemma (3.7), we find

$$\|P(-p^h_\varepsilon/\nu)\|_{H^1(\Gamma)} \le C \quad \text{and} \quad \|\Pi_h(P(-p^h_\varepsilon/\nu))\|_{H^1(\Gamma)} \le C$$

with a positive constant C, independent of ε and h. Using a standard inverse estimate for the first term in (3.18), we continue with

$$\begin{aligned}\|P_h(p^h_\varepsilon)-\Pi_h(P(p^h_\varepsilon))\|_{H^1(\Gamma)} &\le ch^{-1}\|P_h(p^h_\varepsilon)-\Pi_h(P(p^h_\varepsilon))\|_{L^2(\Gamma)}\\ &\le ch^{-1}\big(\|P_h(p^h_\varepsilon)-P(p^h_\varepsilon)\|_{L^2(\Gamma)}\\ &\quad+\|\Pi_h(P(p^h_\varepsilon))-P(p^h_\varepsilon)\|_{L^2(\Gamma)}\big)\end{aligned}$$

Thanks to the approximation error estimate (3.5) and (3.16) in Lemma 3.10, one derives

$$\|P_h(p^h_\varepsilon)-\Pi_h(P(p^h_\varepsilon))\|_{H^1(\Gamma)} \le C$$

with some constant C, independent of ε and h. In Conclusion, we obtain the uniform boundedness of $\bar u^h_\varepsilon$ in $H^1(\Gamma)$. □

REMARK 3.12. *We mention that, for the case $\Omega \subset \mathbb{R}^2$ and Ω convex and polygonally bounded, a proof for the stability of $\bar u^h_\varepsilon$ in $H^1(\Gamma)$ can also be found in a work of Casas and Raymond, see [5]. The underlying analysis is based on arguments that completely differ from the ones used above and only allow to consider the two-dimensional case.*

4. Construction of feasible solutions.

4.1. Multiplier-free optimality conditions. In this section, we derive optimality conditions for the problems (P) and (P_h^ε) respectively, where no Lagrange multiplier occurs. To this end, the admissible sets for problem (P) and (P_h^ε) respectively, are now defined by

$$U_{ad} = \{u \in L^2(\Gamma) \,|\, u_a \le u \le u_b \text{ a.e. on } \Gamma;\ (S\tau^* u)(x) \ge y_c(x) \text{ a.e. in } \Omega'\}$$

and

$$\begin{aligned} V_{ad}^{\varepsilon,h} := \{(u_\varepsilon^h, v_\varepsilon^h) \in U_h \times L^2(\Omega) \,|\, u_a \le u_\varepsilon^h(x) \le u_b \text{ a.e. on } \Gamma,\\ S(\tau^* u_\varepsilon^h + \phi(\varepsilon) E_H^* v_\varepsilon^h)(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon^h(x) \text{ a.e. in } \Omega'\}. \end{aligned}$$

The necessary and sufficient optimality conditions for both problems are formulated in the following lemma.

Lemma 4.1. *Let $(\bar y, \bar u)$ and $(\bar y_\varepsilon^h, \bar u_\varepsilon^h, \bar v_\varepsilon^h)$ be the optimal solutions of problem* (P) *and* (P_h^ε), *respectively. The optimality conditions are given by*

$$(\tau \bar p + \nu \bar u, u - \bar u)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U_{ad}, \tag{4.1}$$

and

$$(\tau \bar p_\varepsilon^h + \nu \bar u_\varepsilon^h, u - \bar u_\varepsilon^h)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar p_\varepsilon^h + \psi(\varepsilon) \bar v_\varepsilon^h, v - \bar v_\varepsilon^h)_{L^2(\Omega)} \ge 0 \quad \forall (u,v) \in V_{ad}^{\varepsilon,h}, \tag{4.2}$$

with the associated adjoint states $\bar p = S^*(\bar y - y_d)$ and $\bar p_\varepsilon^h = S^*(\bar y_\varepsilon^h - y_d)$.

Note that $\bar p$ and $\bar p_\varepsilon^h$, respectively, differ from the adjoint states p and p_ε^h as defined above, since no Lagrange multipliers occur in the right hand side of the respective adjoint equations. Moreover, the variational inequalities in the regularized case cannot be decoupled as in (2.20)-(2.23) and (3.8)-(3.12). The following estimate is the basis for our final error estimate, presented in Section 5.

Lemma 4.2. *For all $u^\delta \in U_{ad}$ and $(u_h^\sigma, 0) \in V_{ad}^{\varepsilon,h}$ there holds*

$$\begin{aligned} \nu \|\bar u - \bar u_\varepsilon^h\|_{L^2(\Gamma)}^2 + \|\bar y - \bar y_\varepsilon^h\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2} \|\bar v_\varepsilon^h\|_{L^2(\Omega)}^2 \\ \le (\tau \bar p + \nu \bar u\,,\, u^\delta - \bar u_\varepsilon^h)_{L^2(\Gamma)} + (\tau \bar p_\varepsilon^h + \nu \bar u_\varepsilon^h\,,\, u_\sigma^h - \bar u)_{L^2(\Gamma)} + C \frac{(\phi(\varepsilon))^2}{\psi(\varepsilon)} \end{aligned} \tag{4.3}$$

for a certain constant $C > 0$ independent of h and ε.

Proof. We start with the variational inequalities of (P) and (P_h^ε) for $u := u^\delta$ and $(u,v) := (u_h^\sigma, 0)$ given by in (4.1) and (4.2), respectively. Adding both inequalities yields

$$\begin{aligned} \nu \|\bar u - \bar u_\varepsilon^h\|_{L^2(\Gamma)}^2 + \psi(\varepsilon) \|\bar v_h^\varepsilon\|^2 \le (\tau \bar p + \nu \bar u\,,\, u^\delta - \bar u_\varepsilon^h)_{L^2(\Gamma)} + (\tau \bar p_\varepsilon^h + \nu \bar u_\varepsilon^h\,,\, u_\sigma^h - \bar u)_{L^2(\Gamma)} \\ + (\tau(\bar p - \bar p_\varepsilon^h)\,,\, \bar u_\varepsilon^h - \bar u)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar p_\varepsilon^h\,,\, -\bar v_\varepsilon^h)_{L^2(\Omega)} \end{aligned}$$

for all $u^\delta \in U_{ad}$ and $(u_h^\sigma, 0) \in V_{ad}^{\varepsilon,h}$. Due to the definitions of the respective states and adjoint states , we continue with

$$\begin{aligned} (\tau(\bar p - \bar p_\varepsilon^h)\,,\, \bar u_\varepsilon^h - \bar u)_{L^2(\Gamma)} &= (\bar y - \bar y_\varepsilon^h\,,\, S\tau^*(\bar u_\varepsilon^h - \bar u))_{L^2(\Omega)} \\ &= (\bar y - \bar y_\varepsilon^h\,,\, \bar y_\varepsilon^h - \bar y)_{L^2(\Omega)} - (\bar y - \bar y_\varepsilon^h\,,\, S\phi(\varepsilon) E_H^* \bar v_\varepsilon^h)_{L^2(\Omega)} \\ &= -\|\bar y - \bar y_\varepsilon^h\|_{L^2(\Omega)}^2 - (E_H(\bar p - \bar p_\varepsilon^h)\,,\, \phi(\varepsilon) \bar v_\varepsilon^h)_{L^2(\Omega)}, \end{aligned}$$

where we again considered S as an operator from $H^1(\Omega)^*$ to $L^2(\Omega)$. Summarizing the terms, we derive

$$\begin{aligned} \nu \|\bar u - \bar u_\varepsilon^h\|_{L^2(\Gamma)}^2 + \|\bar y - \bar y_\varepsilon^h\|_{L^2(\Omega)}^2 + \psi(\varepsilon) \|\bar v_\varepsilon^h\|_{L^2(\Omega)}^2 \\ \le (\tau \bar p + \nu \bar u\,,\, u^\delta - \bar u_\varepsilon^h)_{L^2(\Gamma)} + (\tau \bar p_\varepsilon^h + \nu \bar u_\varepsilon^h\,,\, u_\sigma^h - \bar u)_{L^2(\Gamma)} \\ - (E_H \bar p\,,\, \phi(\varepsilon) \bar v_\varepsilon^h)_{L^2(\Omega)} \end{aligned}$$

The last term is estimated by Young's inequality, and we obtain the assertion (4.3) with the constant $C = \frac{1}{2} \|\bar p\|_{L^2(\Omega)}^2$ independent of ε and h. ∎

The previous lemma shows that it is essential to construct feasible controls $u^\delta \in U_{ad}$ and $(u_h^\sigma, 0) \in V_{ad}^{\varepsilon,h}$, that are close to the respective optimal solution of the other problem.

4.2. Estimation of the maximal violations. In this section, we construct feasible controls for the problem (P) and (P$^\varepsilon_h$), respectively. First, we look for feasible controls for the semi-discretized problem (P$^\varepsilon_h$). To this end, we consider the violation of the control ($\Pi_h\bar u, \bar v \equiv 0$) with respect to the mixed control-state-constraints in (P$^\varepsilon_h$). We define the violation function by

$$\begin{aligned} d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)] &:= (y_c - S\tau^*\Pi_h\bar u - SE_H^*\phi(\varepsilon)0 - \xi(\varepsilon)0)_+ \\ &= \max\{0, y_c - S\tau^*\Pi_h\bar u\}. \end{aligned} \tag{4.4}$$

Furthermore, the $L^\infty(\Omega')$-norm of the violation function $d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]$ is called maximal violation of $(\Pi_h\bar u, 0)$ with respect to (P$^\varepsilon_h$).

LEMMA 4.3. *The maximal violation $\|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')}$ of $(\Pi_h\bar u, 0)$ w.r.t. (P$^\varepsilon_h$) is estimated by*

$$\|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')} \le ch^2, \tag{4.5}$$

where the constant $c > 0$ is independent of h and ε.

Proof. Using the triangle inequality and $\bar y = S\tau^*\bar u$, we find

$$\begin{aligned} \|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')} &= \|(y_c - S\tau^*\Pi_h\bar u)_+\|_{L^\infty(\Omega')} \\ &= \|(y_c - S\tau^*\bar u + S\tau^*(\bar u - \Pi_h\bar u))_+\|_{L^\infty(\Omega')} \\ &\le \|(y_c - \bar y)_+\|_{L^\infty(\Omega')} + \|S\tau^*(\bar u - \Pi_h\bar u)\|_{L^\infty(\Omega')}. \end{aligned}$$

Due to the optimality of $\bar y$ for problem (P), the first term vanishes. Moreover, the optimal control belongs to $H^1(\Gamma)$, see (2.19). Thanks to Corollary 2.3, Lemma 2.4 and (3.6), we find for the second term

$$\|S\tau^*(\bar u - \Pi_h\bar u)\|_{L^\infty(\Omega')} \le c\|S\tau^*(\bar u - \Pi_h\bar u)\|_{L^2(\Omega)} \le c\|\bar u - \Pi_h\bar u\|_{H^1(\Gamma)^*} \le ch^2\|\bar u\|_{H^1(\Gamma)},$$

which implies the assertion. ∎

Next, we construct a feasible solution u^σ_h for (P$^\varepsilon_h$), depending on the inner point of Assumption 2.5 and the optimal solution $\bar u$ of problem (P).

LEMMA 4.4. *Let the Assumption 2.5 be satisfied. Then there is a $\sigma_h \in (0,1)$ so that $(u^\sigma_h := (1-\sigma)\Pi_h\bar u + \sigma\Pi_h\hat u, 0)$ is feasible for (P$^\varepsilon_h$) for all $\sigma \in [\sigma_h, 1]$ and all sufficiently small mesh sizes $h > 0$.*

Proof. Since the operator Π_h, defined in (3.2), preserves the feasibility w.r.t. the control constraints in (P) and (P$^\varepsilon_h$), the convex linear combination $u^\sigma_h := (1-\sigma)\Pi_h\bar u + \sigma\Pi_h\hat u$ fulfills the box constraints. Consequently, we only have to verify

$$y^h_\sigma - y_c \ge 0 \quad \text{a.e. in } \Omega'$$

with

$$y^h_\sigma = S\tau^*u^\sigma_h = (1-\sigma)S\tau^*\Pi_h\bar u + \sigma S\tau^*\Pi_h\hat u.$$

Using the violation function (4.4) and Lemma 3.6, one obtains

$$\begin{aligned} y^h_\sigma - y_c &= (1-\sigma)(S\tau^*\Pi_h\bar u - y_c) + \sigma(S\tau^*\Pi_h\hat u - y_c) \\ &\ge -(1-\sigma)d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)] + \sigma\gamma_0 \\ &\ge -(1-\sigma)\|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')} + \sigma\gamma_0 \end{aligned}$$

for sufficiently small mesh sizes h. Hence, we obtain $(u^\sigma_h, 0) \in V^{\varepsilon,h}_{ad}$ for

$$\sigma \ge \sigma_h := \frac{\|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')}}{\|d[(\Pi_h\bar u, 0), (\mathrm{P}^\varepsilon_h)]\|_{L^\infty(\Omega')} + \gamma_0} \tag{4.6}$$

and sufficiently small mesh sizes $h > 0$. ∎

In the next step, we construct a feasible solution for (P) based on the optimal control $\bar u^h_\varepsilon$ and the inner point $\hat u$. Here, we consider the violation of $\bar u^h_\varepsilon$ w.r.t. the pure state constraints of (P). Now, the violation function is defined by

$$d[\bar u^h_\varepsilon, (P)] := (y_c - S\tau^*\bar u^h_\varepsilon)_+. \tag{4.7}$$

First, we state an auxiliary result, which is important for the estimation of the maximal violation of $\bar{u}_\varepsilon^h$ w.r.t. problem (P).
LEMMA 4.5. *[20, Lemma 3.2] Let f be a uniformly bounded function in $C^{0,1}(\bar{\Omega})$, then there exist a positive constant $c > 0$ such that*

$$\|f\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}^{2/(2+d)}.$$

LEMMA 4.6. *The maximal violation $\|d[\bar{u}_\varepsilon^h, (P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_\varepsilon^h$ w.r.t. problem* (P) *can be estimated by*

$$\|d[\bar{u}_\varepsilon^h, (P)]\|_{L^\infty(\Omega')} \le c\,(\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\, \|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^{2/(2+d)}, \tag{4.8}$$

where the constant $c > 0$ is independent of ε and h.
Proof. According to Sobolev embeddings and (2.6) in Corollary 2.3, we obtain $S\tau^*\bar{u}_\varepsilon^h \in C^{0,1}(\bar{\Omega}')$ and

$$\|S\tau^*\bar{u}_\varepsilon^h\|_{C^{0,1}(\bar{\Omega}')} \le c\|S\tau^*\bar{u}_\varepsilon^h\|_{W^{2,\infty}(\Omega')} \le c\|\bar{u}_\varepsilon^h\|_{L^2(\Gamma)}.$$

Hence, in view of $y_c \in C^{0,1}(\bar{\Omega}')$, the violation function $d[\bar{u}_\varepsilon^h, (P)]$ belongs to $C^{0,1}(\bar{\Omega}')$ and is uniformly bounded with respect to ε and h. We proceed with Lemma 4.5 and obtain

$$\begin{aligned}
\|(y_c - S\tau^*\bar{u}_\varepsilon^h)_+\|_{L^\infty(\Omega')} &\le c\|(y_c - S\tau^*\bar{u}_\varepsilon^h)_+\|_{L^2(\Omega')}^{2/(2+d)} \\
&\le c\left(\|(y_c - S\tau^*\bar{u}_\varepsilon^h - SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon^h)_+\|_{L^2(\Omega')} + \|(SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon^h)_+\|_{L^2(\Omega')}\right)^{2/(2+d)} \\
&= c\left(\|(y_c - \bar{y}_\varepsilon^h)_+\|_{L^2(\Omega')} + \|(SE_H^*\phi(\varepsilon)\bar{v}_\varepsilon^h)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}
\end{aligned}$$

The optimality of $(\bar{y}_\varepsilon^h,\, \bar{u}_\varepsilon^h,\, \bar{v}_\varepsilon^h)$ for $(\mathrm{P}_h^\varepsilon)$ and the continuity of the solution operator yields

$$\begin{aligned}
\|d[\bar{u}_\varepsilon^h, (P)]\|_{L^\infty(\Omega')} \le& c\left(\|(\xi(\varepsilon)\bar{v}_\varepsilon^h)_+\|_{L^2(\Omega)} + \phi(\varepsilon)\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}\right)^{2/(2+d)} \\
&\le c\,(\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\, \|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^{2/(2+d)}.
\end{aligned}$$

This completes the proof. ◻
Now, we can construct a feasible control for the original problem (P).
LEMMA 4.7. *Let Assumption 2.5 be satisfied. Then there exists a $\delta_\varepsilon \in (0, 1)$ such that $u_\delta := (1-\delta)\bar{u}_\varepsilon^h + \delta\hat{u}$ is feasible for problem* (P) *for all $\delta \in [\delta_\varepsilon, 1]$.*
Proof. One can easily see, that the convex linear combination u_δ satisfies the control constraints of problem (P). Consequently, we have to verify the state constraints:

$$\begin{aligned}
y_\delta - y_c &= S\tau^* u^\delta - y_c \\
&= (1-\delta)(S\tau^*\bar{u}_\varepsilon^h - y_c) + \delta(\hat{y} - y_c) \\
&\ge -(1-\delta)(d[\bar{u}_\varepsilon^h, (P)]) + \delta\gamma,
\end{aligned}$$

where we use Assumption 2.5. With this estimate, we derive that $u_\delta \in U_{ad}$ if

$$\delta \ge \delta_\varepsilon := \frac{\|d[\bar{u}_\varepsilon^h, (\mathrm{P})]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_\varepsilon^h, (\mathrm{P})]\|_{L^\infty(\Omega')} + \gamma}, \tag{4.9}$$

which implies the assertion. ◻

5. Main result. In this section, we provide the main result of this paper. We derive error estimate for the L^2-error between the optimal solution of problem (P) and the optimal solution of the semi-discretized problem $(\mathrm{P}_h^\varepsilon)$.
THEOREM 5.1. *Let $(\bar{y},\, \bar{u})$ and $(\bar{y}_\varepsilon^h,\, \bar{u}_\varepsilon^h,\, \bar{v}_\varepsilon^h)$ be the optimal solution of* (P) *and* $(\mathrm{P}_h^\varepsilon)$*, respectively. Then, there exists a constant independent of the meshsize h and ε such that*

$$\begin{aligned}
&\frac{\nu}{2}\,\|\bar{u} - \bar{u}_\varepsilon^h\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|\bar{y} - \bar{y}_\varepsilon^h\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^2 \\
&\qquad \le c\left((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^{2/(2+d)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2\right)
\end{aligned} \tag{5.1}$$

is satisfied for sufficiently small mesh sizes $h > 0$.

Proof. The basis for the proof is the estimate given in Lemma 4.2. Thus, we start with choosing $u_\delta \in U_{ad}$, as defined in Lemma 4.7, with the specific parameter $\delta := \delta_\varepsilon$, given in (4.9). Moreover, the estimate

$$\delta_\varepsilon \le c\|d[\bar{u}_\varepsilon^h, (\mathrm{P})]\|_{L^\infty(\Omega')}$$

is valid for all constants $c \ge 1/\gamma$. Due to (4.8), we derive

$$\begin{aligned}(\tau\bar{p} + \nu\bar{u}\,,\, u^\delta - \bar{u}_\varepsilon^h)_{L^2(\Gamma)} &\le \delta_\varepsilon\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}\|\hat{u} - \bar{u}_\varepsilon^h\|_{L^2(\Gamma)}\\ &\le c\|d[\bar{u}_\varepsilon^h, (P)]\|_{L^\infty(\Omega')}\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}|\Gamma||u_b - u_a|\\ &\le c((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^{2/(2+d)}.\end{aligned} \tag{5.2}$$

Because of optimality, the term $\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}$ can be bounded by expressions containing only data of problem (P). We proceed with the choice $(u_h^\sigma, 0) \in V_{ad}^{\varepsilon,h}$ given by Lemma 4.4 for $\sigma := \sigma_h$, defined in (4.6). Similar to the estimate of δ_ε above, we obtain with (4.5)

$$\sigma_h \le c\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}_h^\varepsilon)]\|_{L^\infty(\Omega')} \le ch^2$$

for sufficiently small mesh sizes h. Hence, we continue with

$$\begin{aligned}(\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} =&\, \sigma_h(\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\,,\, \Pi_h\hat{u} - \Pi_h\bar{u})_{L^2(\Gamma)} +\\ &(\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\,,\, \Pi_h\bar{u} - \bar{u})_{L^2(\Gamma)}\\ \le&\, ch^2\|\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\|_{L^2(\Gamma)}\|\Pi_h(\hat{u} - \bar{u})\|_{L^2(\Gamma)} +\\ &\|\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\|_{H^1(\Gamma)}\|\Pi_h\bar{u} - \bar{u}\|_{H^1(\Gamma)^*}\end{aligned}$$

By Corollary 2.13 and Lemma 3.11, $\bar{p}_\varepsilon^h$ and $\bar{u}_\varepsilon^h$ are bounded by constants independent of h and ε in $H^1(\Gamma)$. The boundedness of the term $\|\Pi_h(\hat{u} - \bar{u})\|_{L^2(\Gamma)}$ follows from Lemma 3.5. Thanks to the approximation error estimate (3.6), we end up with

$$(\tau\bar{p}_\varepsilon^h + \nu\bar{u}_\varepsilon^h\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} \le ch^2. \tag{5.3}$$

By inserting (5.2) and (5.3) in (4.3), we obtain the assertion. □

The result of the previous theorem shows that an L^2-estimate of the virtual control is necessary for completion. In connection with this, we require the following assumption on the coupling of the mesh size h and the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$.

Assumption 5.2. *The parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ are chosen such that*

$$\frac{\phi(\varepsilon) + \xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \sim h^{1+d}. \tag{5.4}$$

Corollary 5.3. *Let the Assumption 5.2 be satisfied. Then, there exist a constant $c > 0$, independent of h and ε, such that*

$$\sqrt{\psi(\varepsilon)}\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)} \le ch \tag{5.5}$$

is valid for sufficiently small mesh sizes h.

Proof. Considering (5.1), we obtain

$$\frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^2 \le c\left((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^{2/(2+d)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2\right)$$

for sufficiently small mesh sizes h. Due to Assumption 5.2, we infer

$$\frac{\psi(\varepsilon)}{2}\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)}^2 \le c\left(h^{\frac{2(1+d)}{2+d}}(\sqrt{\psi(\varepsilon)}\|\bar{v}_\varepsilon^h\|_{L^2(\Omega)})^{\frac{2}{2+d}} + h^2\right).$$

Moreover, this estimate implies

$$\psi(\varepsilon)\|\bar v_\varepsilon^h\|^2_{L^2(\Omega)} \le 2c\max\left\{h^{\frac{2(1+d)}{2+d}}(\sqrt{\psi(\varepsilon)}\|\bar v_\varepsilon^h\|_{L^2(\Omega)})^{\frac{2}{2+d}}, h^2\right\}.$$

We continue by considering the two cases, where the maximum is attained.
Case1: We start with assuming that the maximum is given by h^2. This implies the estimate

$$\|\bar v_\varepsilon^h\|_{L^2(\Omega)} \le c\frac{h}{\sqrt{\psi(\varepsilon)}}.$$

Case2: Now, we assume that the maximum is attained by the first term. Consequently, we find

$$\|\bar v_\varepsilon^h\|^{\frac{2(1+d)}{2+d}}_{L^2(\Omega)} \le ch^{\frac{2(1+d)}{2+d}}(\psi(\varepsilon))^{-\frac{(1+d)}{2+d}}$$
$$\|\bar v_\varepsilon^h\|_{L^2(\Omega)} \le c\frac{h}{\sqrt{\psi(\varepsilon)}}.$$

Summarizing, in both cases, we end up with the same order of convergence with respect to the mesh size h. Hence, the assertion is proven. ∎
Now, we are in the position to state the final error estimate. It immediately follows from Theorem 5.1 and Corollary 5.3.
THEOREM 5.4. *Let* $(\bar y, \bar u)$ *and* $(\bar y_\varepsilon^h, \bar u_\varepsilon^h, \bar v_\varepsilon^h)$ *be the optimal solution of* (P) *and* (P_h^ε)*, respectively. Moreover, let the Assumption 5.2 be satisfied. Then, there exist a positive constant c, independent of* ε *and* h*, such that*

$$\|\bar u - \bar u_\varepsilon^h\|_{L^2(\Gamma)} + \|\bar y - \bar y_\varepsilon^h\|_{L^2(\Omega)} \le ch \tag{5.6}$$

is fulfilled provided that the mesh size h is sufficiently small.
With this result we investigated the error arising from the regularization of problem (P) and the discretization of the boundary control. In the second part [19] of this work we incorporate the finite element discretization of the arising PDEs into the a priori error analysis.

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